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NOTIONS OF INDEPENDENCE FOR RANDOM VARIABLES

BY

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Abstract. The paper considers two known notions of independence for random variables in the probability space $(X, \mathscr{B}, P)$, where X is a subset of the fixed uncountable standard space and $\mathscr{B}$ – the σ-field of Borel subsets relativized to X.

0. Introduction. When are two random variables independent? There are at least two definitions of the concept: in their measure-theoretic form, they may be traced back to "founding fathers" Kolmogoroff and Steinhaus. In response to a challenge of E. Marczewski, both Doob and Jessen [2], [3] simultaneously produced examples to show that, in general, the two definitions are not equivalent. However, with the natural assumption that the underlying probability space is perfect, Steinhaus- and Kolmogoroff-independence come to mean the same. Particulars may be found in [7] and [8].

In [7], D. Ramachandran showed that a certain set-theoretic condition for measurable structures (the strong Blackwell property) also reconciles the two definitions. He asked (P 930) whether a weaker condition (the Blackwell property) would suffice. An example in [13] shows that, at least under the continuum hypothesis (CH), the answer is "no". He also asked (P 931) whether the equivalence of the two notions of independence implies an almost sure Blackwell property. In this paper, we see that the answer is again in the negative (CH). However, for a certain class of singular spaces, a limited form of such a result is available (Proposition 2).

Our main technique in this is the idea of a density of sets relative to a given σ-ideal of Borel sets [10], [12]. Density with respect to the σ-ideal of countable sets is closely related to "Blackwell properties" (Lemma 5), whereas density for the σ-ideal of probability zero sets bears on the question of independence (Proposition 1).

1. Preliminary survey. We deal exclusively with *separable spaces*: these are measurable spaces $(X, \mathscr{B})$ whose σ-algebra $\mathscr{B}$ is countably generated (c.g.)

and contains all singleton sets drawn from X. If A is a subset of X, then A becomes a separable space under the relative structure $\mathscr{B}(A) = \{B \cap A: B \in \mathscr{B}\}$. A separable space $(S, \mathscr{B}(S))$ is *standard* if there is a complete separable metric on S for which $\mathscr{B}(S)$ is the corresponding Borel structure. Any two uncountable standard spaces are Borel-isomorphic, and any separable space is isomorphic with a subset of an uncountable standard space. For these and other customaries concerning separable spaces, we refer the reader to the references [1], [4], or the first parts of [5].

Hereon, the symbol S will denote a fixed uncountable standard space with Borel structure $\mathscr{B} = \mathscr{B}(S)$.

LEMMA 1. *Let X be a subset of S and let $\mathscr{C}(X)$ be a c.g. sub-σ-algebra of $\mathscr{B}(X)$. Then there is a c.g. sub-σ-algebra $\mathscr{C}$ of $\mathscr{B}(S)$ whose relative structure on X is $\mathscr{C}(X)$.*

Proof. Since $\mathscr{C}(X)$ is c.g., there is a real function f on X such that $\mathscr{C}(X) = \{f^{-1}(B)$: B a linear Borel set$\}$. For this technique, consult [1] or [6]. Now there is an extension g of f to all of S which is $\mathscr{B}(S)$-measurable ([4], p. 434, or [9]). Then $\mathscr{C} = \{g^{-1}(B)$: B linear Borel set$\}$ is the desired σ-algebra, q.e.d.

Let I be a σ-ideal in this Borel structure $\mathscr{B}(S)$. A subset R of $S \times S$ in *I-reticulate* if there is some set N in I with $R \subset (N \times S) \cup (S \times N)$. A subset X of S is *I-dense* (*of order* 1) if X intersects every set in $\mathscr{B}(S) \backslash I$. A subset X of S is *I-dense of order* 2 if $X \times X$ intersects every set R in $\mathscr{B}(S \times S)$ which is not I-reticulate. We shall be interested in two particular σ-ideals.

Let $I(c)$ be the σ-ideal of all countable subsets of S. In keeping with the phraseology of earlier work [10], [12], we shall use the term *Borel-dense* to mean $I(c)$dense. Let m be a Borel probability measure on S. Define $I(m)$ to be the σ-ideal of all m-null members of $\mathscr{B}(S)$. Note that X is $I(m)$-dense in S if and only if $m^*(X) = 1$.

By a *probability space* we mean a triple $(X, \mathscr{B}(X), P)$, where P is a probability measure on $\mathscr{B}(X)$ and $(X, \mathscr{B}(X))$ is a separable space. Suppose that X is a subset of S. Then each probability P on X gives rise to a probability $\bar{P}$ on S. Define $\bar{P}(B) = P(B \cap X)$ for B in $\mathscr{B}(S)$ to be the probability *induced* by P. We may pass freely between probabilities on S and on $X \subset S$ *via* the easy

LEMMA 2. *Let m be a probability on S and let P be a probability on $X \subset S$. Then*

(a) $(\bar{P})^* = P$;

(b) *if* $m^*(X) = 1$, *then* $\overline{m^*} = m$.

A probability space $(X, \mathscr{B}, P)$ is *perfect* if there is some standard set $B \in \mathscr{B}(X)$ with $P(B) = 1$. This definition differs from the usual one, but the two agree when $(X, \mathscr{B}(X))$ is separable ([8], 2.4.1, and [11], Theorem 2). Say that a separable space X is *universally measurable* (u.m.) if $(X, \mathscr{B}(X), P)$ is perfect for every probability P on X. Again, this is different from the usual definition, but the two coincide for separable spaces ([11], Lemma 4). Our perfect spaces are those termed "metrically standard" by Mackey [5].

Let m be a probability on S. Two sub-σ-algebras $\mathscr{C}$ and $\mathscr{D}$ in $\mathscr{B}(S)$ are *m-independent* if $m(C \cap D) = m(C)m(D)$ for all $C \in \mathscr{C}$ and $D \in \mathscr{D}$. A real measurable function ξ on a probability space $(X, \mathscr{B}, P)$ is called a *random variable*. Each random variable generates two "spectral" σ-algebras in $\mathscr{B}(X)$:

$\mathscr{B}(\xi) = \{\xi^{-1}(B)$: B linear Borel set$\}$;

$\mathscr{A}(\xi) = \{\xi^{-1}(A) \in \mathscr{B}(X)$: A linear set$\}$.

Clearly, $\mathscr{B}(\xi) \subset \mathscr{A}(\xi) \subset \mathscr{B}(X)$. Random variables ξ and η are *Steinhaus-independent* [resp. *Kolmogoroff-independent*] if $\mathscr{B}(\xi)$ and $\mathscr{B}(\eta)$ [resp. $\mathscr{A}(\xi)$ and $\mathscr{A}(\eta)$] are P-independent. Say that $(X, \mathscr{B}, P)$ is an *independence space* if these notions of independence coincide. There are classical examples due to Doob and Jessen [2], [3] to show that not every probability triple is an independence space. Say that a separable $(X, \mathscr{B})$ is a *universal independence space* if for every probability P on X, the triple $(X, \mathscr{B}, P)$ is an independence space. Using Lemma 2, it is easy to establish

LEMMA 3. *Let m be a Borel probability on S and let $X \subset S$ be such that $m^*(X) = 1$. Put $P = m^*$ on $(X, \mathscr{B}(X))$ and suppose that $\mathscr{C}$ and $\mathscr{D}$ are sub-σ-algebras of $\mathscr{B}(S)$. Then the following are equivalent*:

(1) *$\mathscr{C}$ and $\mathscr{D}$ are m-independent*;

(2) *$\mathscr{C}(X)$ and $\mathscr{D}(X)$ are P-independent.*

We say that a separable space $(X, \mathscr{B}(X))$ is *strongly Blackwell* [resp. *Blackwell*] if for each real measurable function [resp. one-one function] ξ on X, one has $\mathscr{B}(\xi) = \mathscr{A}(\xi)$. For a survey of results on Blackwellian properties, see [1]. We mention a few basic facts.

Fact 1. Let $(X, \mathscr{B})$ be a separable space. Then the following are equivalent:

(1) $(X, \mathscr{B})$ is a Blackwell space;

(2) whenever $\mathscr{C}(X)$ is a c.g. sub-σ-algebra of $\mathscr{B}(X)$ separating points of X, then $\mathscr{C}(X) = \mathscr{B}(X)$;

(3) whenever ξ is a one-one measurable real function on $(X, \mathscr{B})$, then ξ is a Borel isomorphism of X onto its image $\xi(X)$.

Fact 2. Let $(X, \mathscr{B})$ be a separable space. Then the following are equivalent:

(1) $(X, \mathscr{B})$ is strongly Blackwell;

(2) whenever $\mathscr{C}(X)$ and $\mathscr{D}(X)$ are c.g. sub-σ-algebras of $\mathscr{B}(X)$ with the same atoms, then $\mathscr{C}(X) = \mathscr{D}(X)$;

(3) whenever ξ is a measurable real function on $(X, \mathscr{B})$, then $\xi(X) \subset \boldsymbol{R}$ has the Blackwell property.

Fact 3. If $(X, \mathscr{B})$ is Blackwell [resp. strong Blackwell], then every member of $\mathscr{B}(X)$ is Blackwell [resp. strong Blackwell] in its relative structure.

Fact 4. Every standard or analytic space is strongly Blackwell.

Fact 5. There is at least one co-analytic space without the Blackwell property.

Fact 6. There is a strong Blackwell space $(X, \mathscr{B})$ which is not u.m.

Proofs of all of these points and a short history of these objects are to be found in [1]. The question of whether there is a Blackwell space not strongly Blackwell is unsettled except under some extra set-theoretic assumptions (such as MA or CH, where it is true — this according to some unpublished work of D. Fremlin, W. Bzyl and J. Jasiński).

Lemma 4. *Every strong Blackwell* $(X, \mathscr{B})$ *is a universal independence space.*

Proof. This follows quickly from the definitions, since $\mathscr{B}(\xi)$ and $\mathscr{A}(\xi)$ always coincide.

In [12], one finds a proof of the following

Lemma 5. *Let X be Borel-dense in S. Then the following are equivalent*:

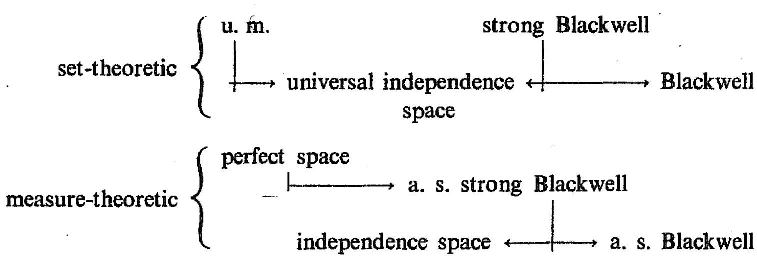
(1) $(X, \mathscr{B}(X))$ *is strong Blackwell*;

(2) $(X, \mathscr{B}(X))$ *is Blackwell*;

(3) X *is Borel-dense of order* 2.

Say that a probability space $(X, \mathscr{B}(X), P)$ is *almost surely* (a.s.) *Blackwell* [resp. *strong Blackwell*] if there is some B in $\mathscr{B}(X)$ with $P(B) = 1$ and $(B, \mathscr{B}(B))$ Blackwell [resp. strong Blackwell].

To summarise, we offer the following diagram of implications:



Properties in the upper bracket are set-theoretic and apply to a separable space $(X, \mathscr{B}(X))$; those in the lower bracket are measure-theoretic and apply to a probability space $(X, \mathscr{B}(X), P)$ with separable Borel structure $\mathscr{B}(X)$. Within each bracket, no other implications may be added to the network. To see this, we recall facts 5 and 6 *supra*, noting that every co-analytic space is u.m. Also:

1. Assuming the continuum hypothesis CH, or Martin's Axiom MA,

there is a Blackwell space which is not a universal independence space. An example is constructed in [13].

2. There is an independence space $(X, \mathscr{B}, P)$ which is not a.s. Blackwell. This is one of the main results of this paper: v. the example in the next section.

2. New results. The imposition of the strong Blackwell property on a separable space provides an easy escape from any paradox involving discrepancy between the Steinhaus and Kolmogoroff definitions of independence. It must be said, however, that it is a rather artificial restriction and, in view of our counter-examples, one that is overly severe. A more natural condition is that of $I(m)$-density of order 2, which we conjecture to be equivalent to the independence property.

LEMMA 6. *Let m be a Borel probability on S and suppose that $X \subset S$ is $I(m)$-dense of order 2 in S. Suppose that $\mathscr{C} \subset \mathscr{C}_1$, are c.g. sub-$\sigma$-algebras of $\mathscr{B}(S)$ such that $\mathscr{C}(X)$ and $\mathscr{C}_1(X)$ have the same atoms. Then there is a set N in $\mathscr{B}(S)$ with $mN = 0$ and $\mathscr{C}(S \backslash N) = \mathscr{C}_1(S \backslash N)$.*

Proof. Let f and f_1 be real functions generating the σ-algebras $\mathscr{C}$ and $\mathscr{C}_1$. Define

$$T = \{(s, t) \in S \times S\colon f(s) = f(t) \text{ and } f_1(s) \neq f_1(t)\}.$$

Then $T \cap (X \times X) = \emptyset$, so that there is some N as indicated with $T \subset (N \times S) \cup (S \times N)$. Then $\mathscr{C}(S \backslash N)$ and $\mathscr{C}_1(S \backslash N)$ have the same atoms. Since $S \backslash N$ is standard, it has the strong Blackwell property. This implies that $\mathscr{C}(S \backslash N) = \mathscr{C}_1(S \backslash N)$, q.e.d.

PROPOSITION 1. *Let m be a Borel probability on a standard space S and let $X \subset S$ be $I(m)$-dense of order 2 in S. Then $(X, \mathscr{B}(X), m^*)$ is an independence space.*

Proof. Let $\mathscr{C}(X)$ and $\mathscr{D}(X)$ be c.g. sub-σ-algebras of $\mathscr{B}(X)$ which are m^*-independent. By Lemma 1, there are c.g. sub-σ-algebras $\mathscr{C}$ and $\mathscr{D}$ of $\mathscr{B}(S)$ whose relativisations to X are $\mathscr{C}(X)$ and $\mathscr{D}(X)$, respectively. From Lemma 3, $\mathscr{C}$ and $\mathscr{D}$ are m-independent.

Suppose that A and B are sets in $\mathscr{B}$ such that $A \cap X$ and $B \cap X$ are unions of $\mathscr{C}(X)$-atoms and $\mathscr{D}(X)$-atoms, respectively. Define $\mathscr{C}_1 = \sigma(\mathscr{C}, A)$ and $\mathscr{D}_1 = \sigma(\mathscr{D}, B)$. We must establish that $\mathscr{C}_1(X)$ and $\mathscr{D}_1(X)$ are m^*-independent.

Using Lemma 6, we produce a set N in $\mathscr{B}(S)$ with $m(N) = 0$ such that $\mathscr{C}(S \backslash N) = \mathscr{C}_1(S \backslash N)$ and $\mathscr{D}(S \backslash N) = \mathscr{D}_1(S \backslash N)$. Calculate:

$$\begin{aligned} m^*(A \cap X \cap B \cap X) &= m(A \cap B) = m(A \cap B \cap N^c) = m(A \cap N^c \cap B \cap N^c) \\ &= m(A \cap N^c)\, m(B \cap N^c) = m(A)\, m(B) \\ &= m^*(A \cap X)\, m^*(B \cap X), \end{aligned}$$

q.e.d.

Conjecture. Let P be a Borel probability on $X \subset S$. Then the following are equivalent:

(1) $(X, \mathscr{B}(X), P)$ is an independence space;

(2) X is $I(\bar{P})$-dense of order 2 in S.

The following example decides (P 931) in [7], which is Q.5 in [8].

Example. Assuming CH, there is a non-perfect independence space which is not a.s. Blackwell.

Construction. We take S to be the unit interval $]0, 1[$ under the usual standard structure and define m to be Lebesgue measure on S. List the uncountable members of $\mathscr{B}(S)$ as $B_0 B_1 B_2 \ldots B_\alpha \ldots, \alpha < c$; list the sets in $\mathscr{B}(S \times S)$ which are not $I(m)$-reticulate as $R_0 R_1 R_2 \ldots R_\alpha \ldots$, $\alpha < c$; finally, list the m-null sets in $\mathscr{B}(S)$ as $N_0 N_1 N_2 \ldots N_\alpha \ldots$, $\alpha < c$. For each $\alpha < c$, define $M_\alpha = \bigcup \{N_\beta \colon \beta \leqslant \alpha\}$.

Define $f\colon S \to S$ by the rule $f(s) = 1 - s$. Define, for each $\alpha < c$, the set $G_\alpha = \{(s, f(s))\colon s \in M_\alpha \text{ or } f(s) \in M_\alpha\}$. The G_α form an increasing transfinite sequence of symmetric sets. We shall construct $X \subset S$ as the union of sets X_α. Define $K_0 = \emptyset$ and $K_\alpha = \bigcup \{X_\beta \colon \beta < \alpha\}$ and choose $x_\alpha \in B_\alpha \setminus f(K_\alpha)$ and $(y_\alpha, z_\alpha) \in R_\alpha \cap \big[(M_\alpha \cup f(M_\alpha))^c \times (M_\alpha \cup f(M_\alpha))^c\big]$. Put $X_\alpha = K_\alpha \cup \{x_\alpha, y_\alpha, z_\alpha\}$ and, finally, $X = \bigcup \{X_\alpha \colon \alpha < c\}$.

Then X is Borel-dense in S and is $I(m)$-dense of order 2 in S. Yet for each α, the set $(X \times X) \cap G_\alpha$ is of cardinality less than c. Thus X is not Borel-dense of order 2 in S and for each $\beta < c$, the set $X \setminus N_\beta$ is Borel-dense in $S \setminus N_\beta$ of order 1, but not of order 2.

Defining $P = m^*$ on X, we see that Proposition 1 implies that $(X, \mathscr{B}(X), P)$ is an independence space. On the other hand, Lemma 5 shows that $(X, \mathscr{B}(X), P)$ is not a.s. Blackwell.

PROPOSITION 2. *Let X be a Borel-dense subset of S. Then the following conditions are equivalent*:

(1) *$(X, \mathscr{B}(X))$ is a universal independence space*;

(2) *$(X, \mathscr{B}(X))$ is strongly Blackwell*;

(3) *$(X, \mathscr{B}(X))$ is Blackwell*;

(4) *X is Borel-dense of order 2 in S.*

Proof. The equivalence of conditions 2, 3, 4 was stated in Lemma 5. The implication $2 \Rightarrow 1$ follows from Lemma 4. It remains only to prove that $1 \Rightarrow 4$. We shall establish the contrapositive. Suppose that X is Borel-dense of order 1, but not of order 2, in S. Then, according to Lemma 5 in [12], there is a Borel-automorphism α of S onto itself such that

(a) $\alpha = \alpha^{-1}$,

(b) the set $T = \{(s, t)\colon \alpha(s) \neq s\}$ is uncountable and does not meet the set $X \times X$.

Define the sets

$$S_0 = \{s \in S: \alpha(s) = s\}, \quad S_1 = \{s \in S: \alpha(s) < s\}, \quad S_2 = \{s \in S: \alpha(s) > s\}.$$

Note that $\alpha(S_0) = S_0$, $\alpha(S_1) = S_2$, and $\alpha(S_2) = S_1$. Let m_1 be a continuous probability on S with $m_1(S_1) = 1$. Let m_2 be the image measure $m_2 = \alpha(m_1) = m_1 \alpha^{-1}$. Define $m = (m_1 + m_2)/2$.

It is no loss of generality to assume that S is some Borel subset of the real line. This we do, defining $f: S \to S$ by the rule $f(s) = s \Lambda \alpha(s)$, the lesser of s and $\alpha(s)$. Define the sub-σ-algebras $\mathscr{C}$ and $\mathscr{D}$ of $\mathscr{B}(S)$ as

$$\mathscr{C} = \mathscr{B}(f) = \{f^{-1}(B): B \in \mathscr{B}(S)\} \text{ and } \mathscr{D} = \sigma(S_0, S_1, S_2).$$

We see that $\mathscr{C}$ and $\mathscr{D}$ are m-independent. For example, let $B \in \mathscr{B}(S)$ and put $B_0 = B \cap S_0$, $B_1 = B \cap S_1$, $B_2 = B \cap S_2$. Then

$$f^{-1}(B) = B_0 \cup \alpha(B_2) \cup B_2,$$

so that $mf^{-1}(B) = \big(m_1 \alpha(B_2) + m_2(B_2)\big)/2 = m_1 \alpha(B_2)$. Also,

$$m\big(f^{-1}(B) \cap S_1\big) = m\alpha(B_2) = m_1 \alpha(B_2)/2 = m\big(f^{-1}(B)\big) m(S_1).$$

Hence, S_1 is independent of $\mathscr{C}$, as are S_0 and S_2.

Since X is Borel-dense in S and m is continuous, one has $m^*(X) = 1$ and the m^*-independence of $\mathscr{C}(X)$ and $\mathscr{D}(X)$. However, $\mathscr{C}(X)$ is separable, so that $X_1 = S_1 \cap X$ is a set in $\mathscr{B}(X)$ which is a union of $\mathscr{C}(X)$-atoms. Now, X_1 belongs to $\mathscr{D}(X)$, and cannot be m^*-independent of itself. This is the desired contradiction, q.e.d.

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